LECTURE 10: l_2 REGULARIZATION

STAT 598z: Introduction to computing for statistics

Vinayak Rao

Department of Statistics, Purdue University

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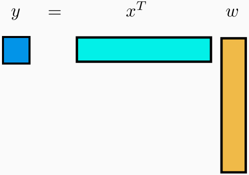
Ordinary least squares

Consider linear regression:

$$y = \mathbf{x}^\top \mathbf{w} + \epsilon$$



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In vector notation:

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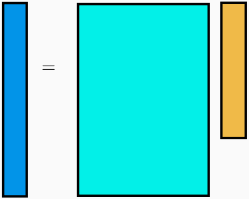

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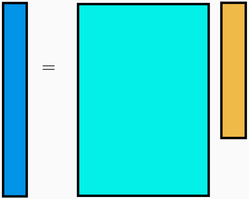

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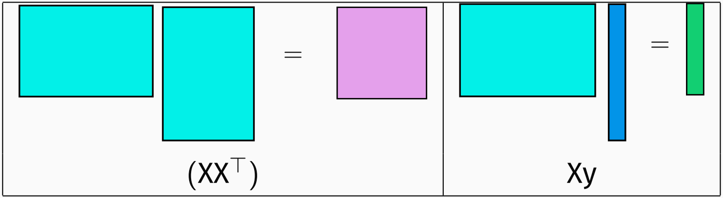

$(\mathrm{X}\mathrm{X}^\top)$

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- Do not invert with `solve` and multiply!

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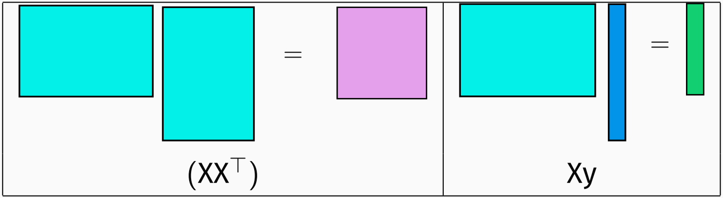

How to do this in `R` (without using `lm`)?

- Do not invert with `solve` and multiply!
- Directly solve $(\mathbf{X}\mathbf{X}^\top)\hat{\mathbf{w}} = \mathbf{X}\mathbf{y}$

Prediction error

$\hat{\mathbf{w}}$ is an unbiased estimate of the true $\mathbf{w}$

For a test vector $\mathbf{x}^{test}$ we predict $\mathbf{w}^\top \mathbf{x}^{test}$.

(Squared) prediction error: $PE^2 = \frac{1}{k}\sum_{i=1}^{k}(y_i^{test} - \mathbf{w}^\top \mathbf{x}_i^{test})^2$

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What if $p > n$?

- $\mathbf{X}\mathbf{X}^\top$ is singular

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λ trades-off bias and variance

Maybe a nonzero λ is actually good?

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ℓ_2/ridge/Tikhonov regularization

Ridge regression (solution)

Simple modification of the least-squares solution:

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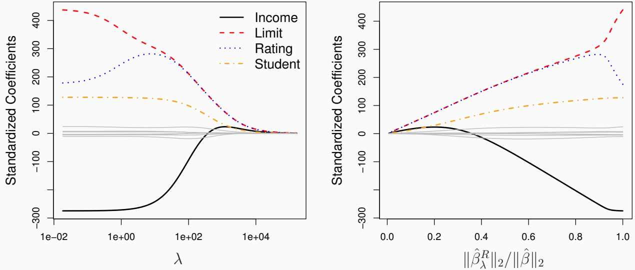
Shrinks least-squares solution.

Ridge regression

Credit data set (average credit card debt)



James, Witten, Hastic and Tibshirani

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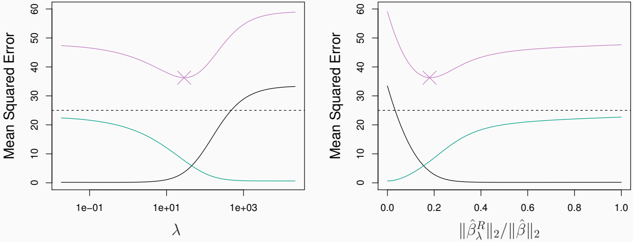
- Having chosen $\hat{\lambda}$ solve regularized least square on all data

Does this work?



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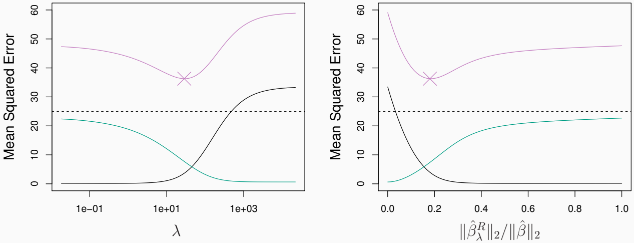

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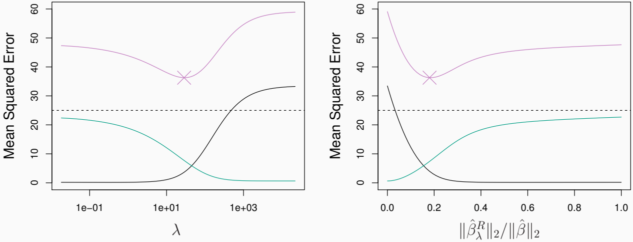

Ridge regression improves performance by reducing variance

- does not perform feature selection
- just shrinks components of $\mathbf{w}$ towards 0

For the former: Lasso

Regularization as constrained optimization

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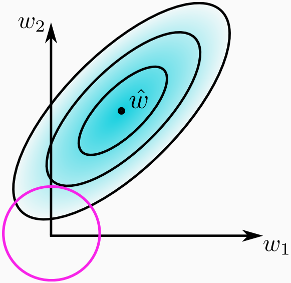

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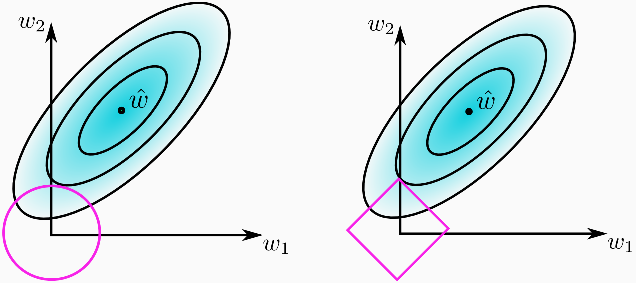
w_2
$\hat{w}$
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