LECTURE 5: COMPLEXITY, DATA-STRUCTURES AND SORTING

STAT 545: Intro. to Computational Statistics

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LINEAR ALGEBRA

Let x and y be $N \times 1$ vectors

Let A and B be $N \times N$ and $N \times M$ matrices

How many additions and multiplications to calculate:

- $x^\top y$
- Ax
- AB
- A^{-1}

BIG-O NOTATION

The big-O notation provides an asymptotic upper bound:

$$O(g(N)) = \{f : \exists c, N_0 > 0 \; s.t. \; f(N) \leq cg(N) \; \forall N > N_0\}$$

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$N^2 \in O(N^3)$
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So is matrix multiplication $O(N^3)$? Yes, but: it's also $O(N^{2.38})$!

Conjecture: matrix multiplication is actually $O(N^2)$.

Sorting

Consider a set of N numbers. We want to sort them in decreasing order. What is the complexity?

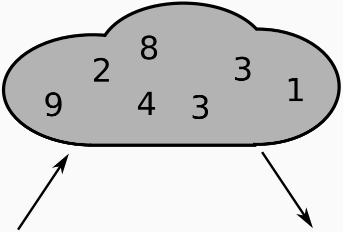
Naïve algorithm:

- Find smallest number. Cost? $O(N)$
- Find next smallest number. Cost? $O(N)$
- . . .

Overall cost? $O(N^2)$
Can we do better?

PRIORITY QUEUE

A ‘bag’ with three commands: `Insert`, `FindMax` and `RemoveMax`.



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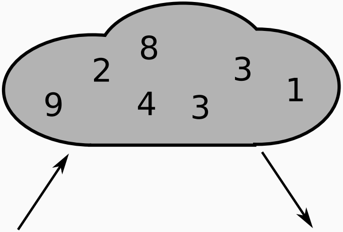

`RemoveMax`

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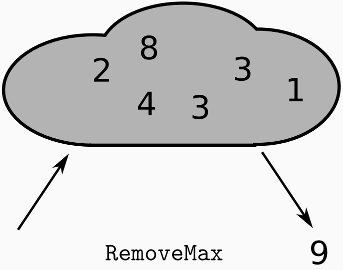

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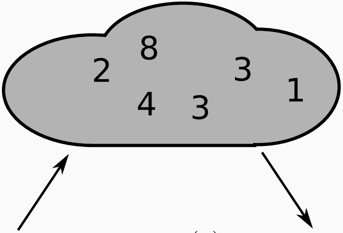

`Insert(6)`

PRIORITY QUEUE

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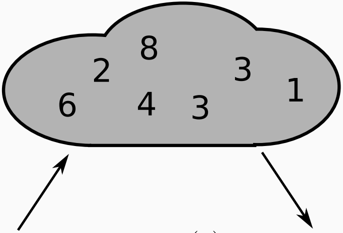

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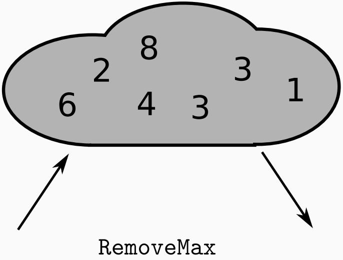

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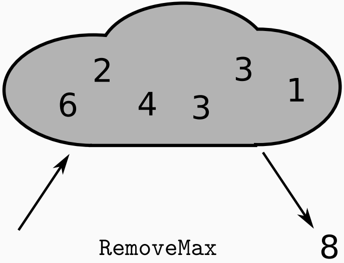

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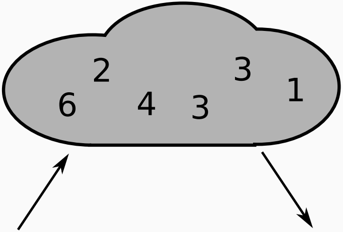

`RemoveMax`

PRIORITY QUEUE

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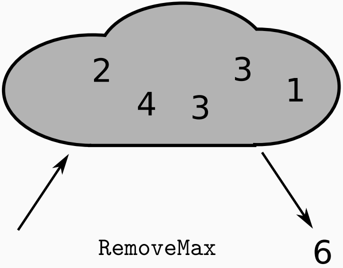

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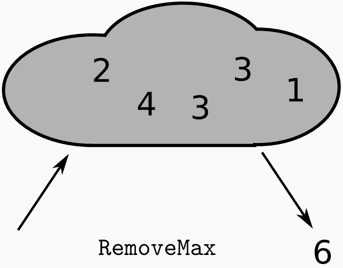
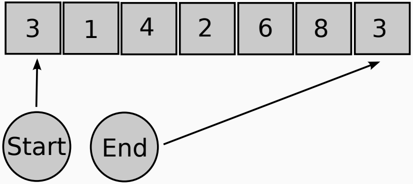


Sorting, clustering, discrete-event simulation, queuing systems
How do we implement this?

PRIORITY QUEUE

Naive approach 1: an unsorted array:

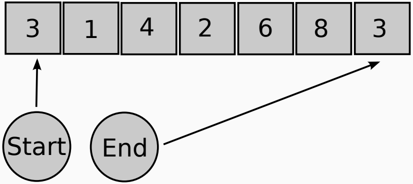
$$(3, 1, 4, 2, 6, 8, 3)$$



Priority queue

Naive approach 1: an unsorted array:

$$(3, 1, 4, 2, 6, 8, 3)$$



What is the cost of `Insert`? $O(1)$

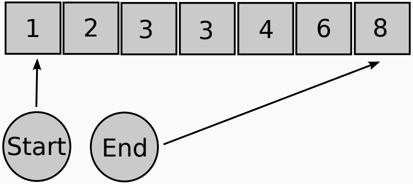
What is the cost of `FindMax`? $O(N)$

What is the cost of `RemoveMax` (assume we've already found the maximum)? $O(1)$

PRIORITY QUEUE

Naive approach 2: a sorted array:

$$(1, 2, 3, 3, 4, 6, 8)$$



PRIORITY QUEUE

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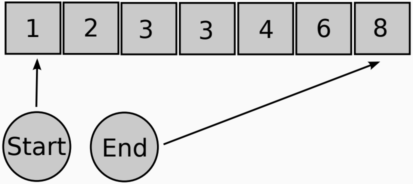


Cost of `FindMax`? $O(1)$

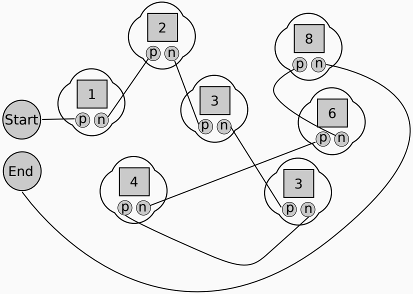
Cost of `RemoveMax`? $O(1)$

Cost of `Insert.FindPosition`? $O(\log(N))$ (binary search)

Cost of `Insert.Insert`? $O(N)$

PRIORITY QUEUE

Naive approach 3: a sorted doubly linked-list:



PRIORITY QUEUE

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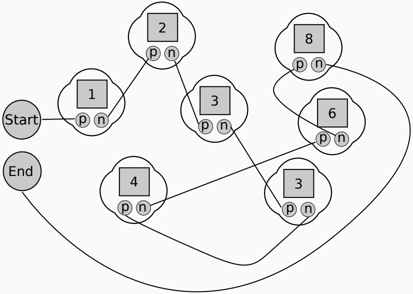

What is the cost of `FindMax`? $O(1)$

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PRIORITY QUEUE

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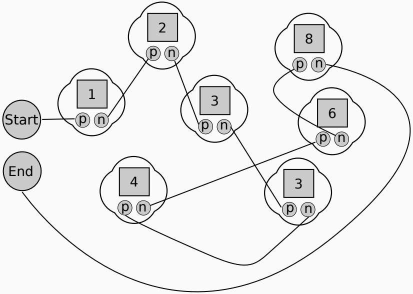


What is the cost of `FindMax`? $O(1)$

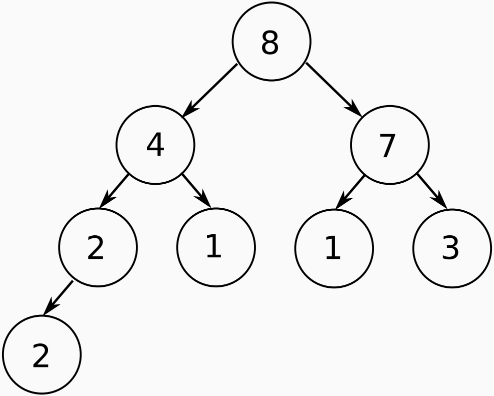
What is the cost of `RemoveMax`? $O(1)$

What is the cost of `Insert`? $O(N)$

Each approach solves one problem, but makes another operation N. Can we do better?

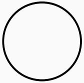
HEAPS

We need a more complicated data-structure: a Heap.



For a precise definition, see: `http://pages.cs.wisc.edu/~vernon/cs367/notes/11.PRIORITY-Q.html`

HEAPS



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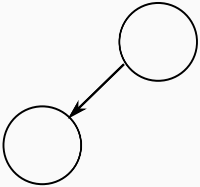

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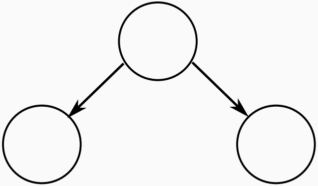

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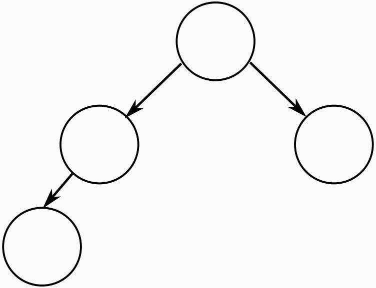

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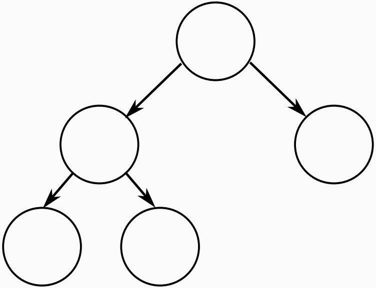

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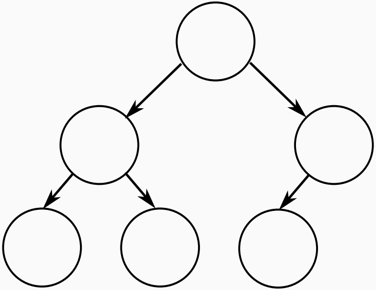

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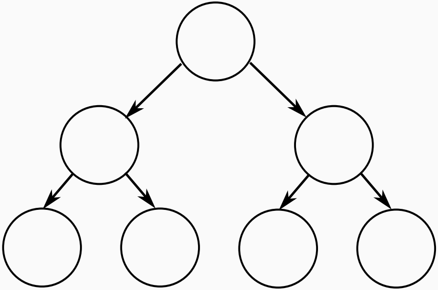

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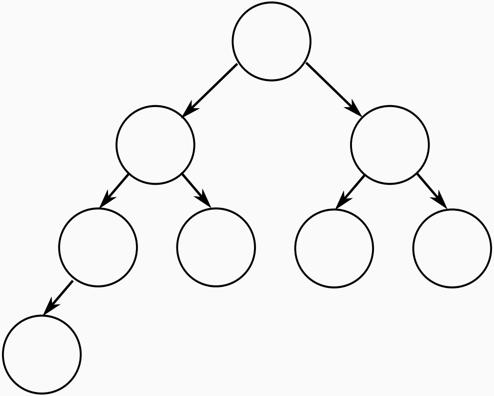

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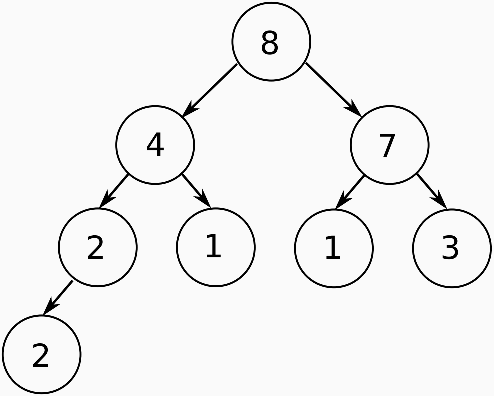

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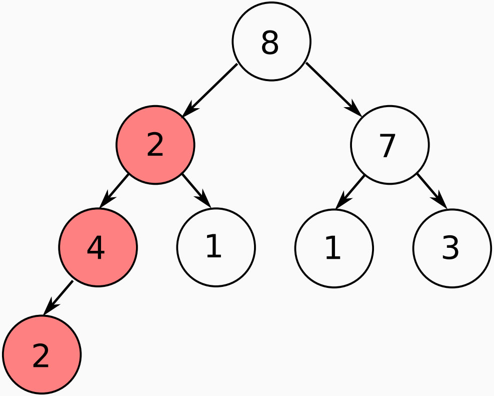

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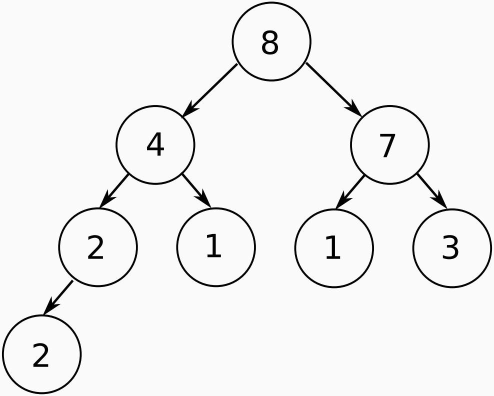
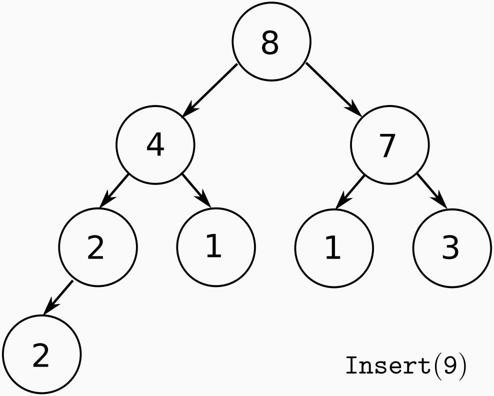
HEAPS: `Insert`

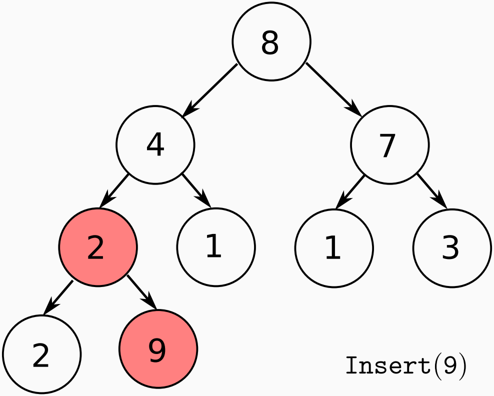


HEAPS: `Insert`



`Insert(9)`

HEAPS: `Insert`



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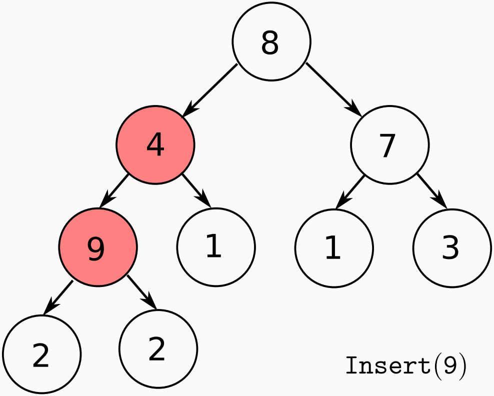
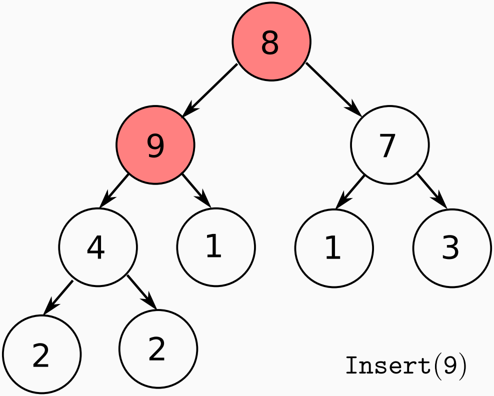
HEAPS: `Insert`



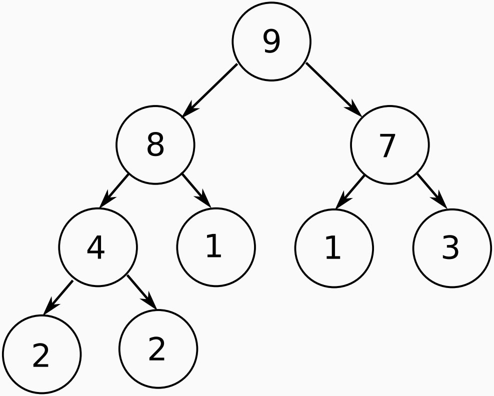
`Insert(9)`

HEAPS: `Insert`

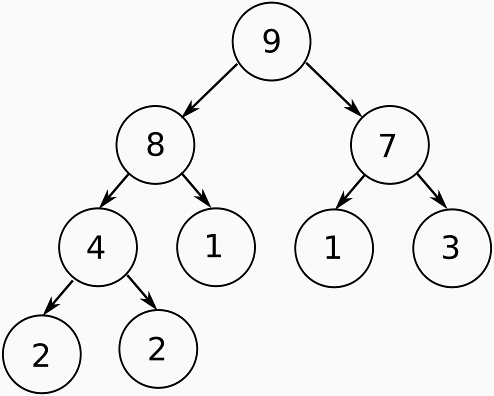


`Insert`(9)

HEAPS: `Insert`

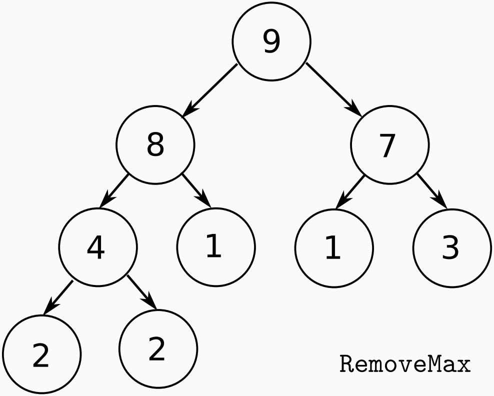


HEAPS: `Insert`



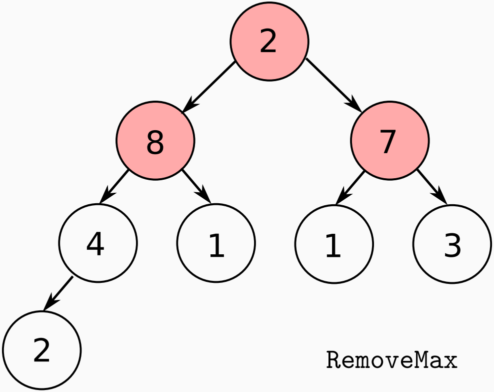
Cost? $O(\log(N))$

HEAPS: `RemoveMax`



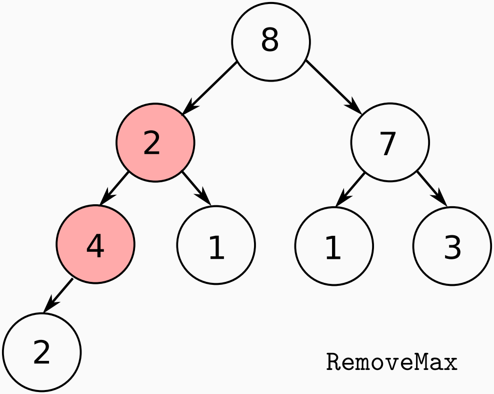
`RemoveMax`

HEAPS: `RemoveMax`



`RemoveMax`

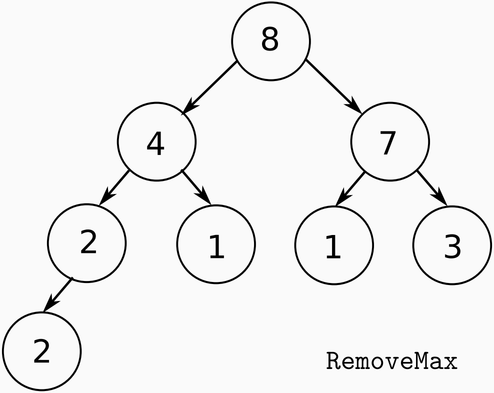
Heaps: `RemoveMax`



`RemoveMax`

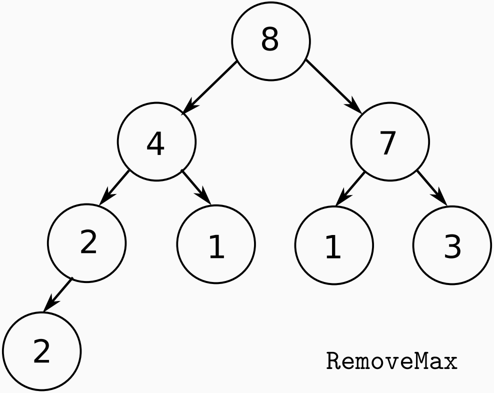
Swap with larger child

HEAPS: `RemoveMax`



`RemoveMax`

HEAPS: `RemoveMax`



`RemoveMax`

Cost? $O(\log(N))$

HEAPSORT

Consider a set of N numbers. Want to sort in decreasing order.

Grow a priority queue, sequentially adding elements

- Cost of each step? $O(\log(N))$
- Overall cost? $O(N\log(N))$

Sequentially remove the maximum element

- Cost of each step? $O(\log(N))$
- Overall cost? $O(N\log(N))$

Cost of overall algorithm? $O(N\log(N))$

QUICKSORT

See `http://me.dt.in.th/page/Quicksort/` for a nicer display

- Choose a pivot
- Ensure all elements to left of pivot are less than/equal, and all to right are greater than the pivot
- Recurse for each half

7	3	2	1	5	9	4

QUICKSORT

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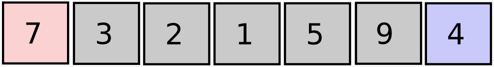

7	3	2	1	5	9	4

- Pivot: #7 (Blue)
- Start: #1 (Red)
- End: #6

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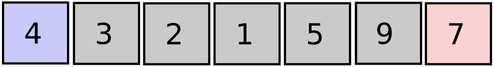

4	3	2	1	5	9	7

- Pivot: #1
- Start: #2
- End: #6

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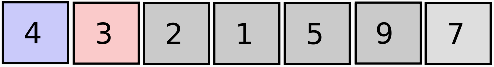

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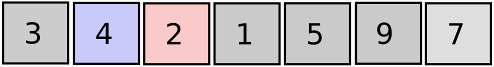

3	4	2	1	5	9	7

- Pivot: #2
- Start: #3
- End: #6

QUICKSORT

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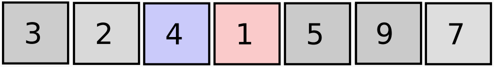

3	2	4	1	5	9	7

- Pivot: #3
- Start: #4
- End: #6

Quicksort

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Recurse for each half

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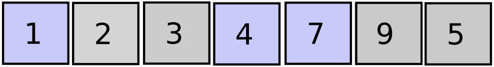

1	2	3	4	7	9	5

Recurse for each half

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See `http://me.dt.in.th/page/Quicksort/` for a nicer display

- Choose a pivot
- Ensure all elements to left of pivot are less than/equal, and all to right are greater than the pivot
- Recurse for each half

1	2	3	4	7	9	5

At the end, we have a sorted list

ANALYSIS OF QUICKSORT

Analysis is a bit harder

What is the worst-case runtime?

What is the best-case runtime?

Average run-time is $\Theta(n \log n)$

Average with respect to what?

Randomized algorithms

FINAL (INFORMAL) COMMENTS

Class **P**: Problems of polynomial complexity. Let $T(n)$ be running-time for input size n. Then there is a k such that:

$$T(n) = O(n^k)$$

Class **E**: Problems of exponential complexity.

$$T(n) = O(\exp(n))$$

Class **NP**: Problems of where proposed solution can be verified in polynomial time. E.g. graph isomorphism
Class **NP**-complete: Hardest problems in **NP** (i.e. problems in both **NP** and **NP**-hard). E.g. travelling salesman.
Class **NP**-hard: at least as hard as the hardest problems in **NP** (halting problem)

P = NP?

A million dollar question (literally)

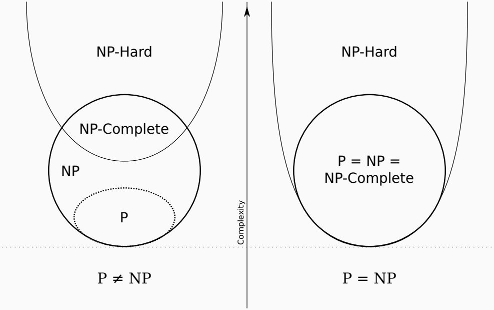

Figure: from Wikipedia