Midterm 1, 2015

Q5) $X = \{x_1 \dots x_N\}$ come from a mixture of 2 Gaussians w. mean 0 & variance σ_1^2, σ_2^2. π is the prob of cluster 1. $C = \{c_1 \dots c_N\}$ are cluster assignments. Derive the EM alg to find σ_1^2, σ_2^2 & π

1. $$P(X, C \mid \pi, \sigma_1^2, \sigma_2^2) = \prod_{i=1}^{N} \left[\pi \cdot \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x_i^2}{2\sigma_1^2}\right)\right]^{\delta(c_i=1)} \left[(1-\pi) \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x_i^2}{2\sigma_2^2}\right)\right]^{\delta(c_i=0)}$$

$$\log P(X, C \mid \pi, \sigma_1^2, \sigma_2^2) = \sum_{i=1}^{N} \Bigg(\delta(c_i=1)\log\pi - \frac{1}{2}\delta(c_i=1)\log 2\pi\sigma_1^2 - \frac{\delta(c_i=1)\, x_i^2}{2\sigma_1^2} + \delta(c_i=2)\log(1-\pi) - \frac{1}{2}\delta(c_i=2)\log 2\pi\sigma_2^2 - \frac{\delta(c_i=2)\, x_i^2}{2\sigma_2^2} \Bigg)$$

2. $$F(q, \sigma_1^2, \sigma_2^2) = \mathbb{E}_q\left[\log P(X, C \mid \pi, \sigma_1^2, \sigma_2^2)\right]$$
$$= \sum_{i=1}^{N} \left(q_i \log\pi - \frac{1}{2} q_i \log 2\pi\sigma_1^2 - \frac{q_i}{2\sigma_1^2} x_i^2 + (1-q_i)\log(1-\pi) - \frac{1}{2}(1-q_i)\log 2\pi\sigma_2^2 - \frac{(1-q_i)\, x_i^2}{2\sigma_2^2} \right)$$

3.
a) ~~For~~ $q_i(c_i=1) := q_i \propto \pi \cdot \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{x_i^2}{2\sigma_1^2}\right)$, $q_i(c_i=0) \propto (1-\pi)\frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{x_i^2}{2\sigma_2^2}\right)$

(You should try this 方法)

b) To find σ_1^2, solve $\frac{\partial F}{\partial \sigma_1^2} = 0$

i.e. $$\sum_i -\frac{1}{2} \cdot \frac{q_i}{\sigma_1^2} + \frac{q_i x_i^2}{2(\sigma_1^2)^2} = 0$$

i.e. $$\sigma_1^2 = \frac{\sum_{i=1}^{N} q_i x_i^2}{\sum_{i=1}^{N} q_i}$$

Try solving for π yourself.