
Supplementary material for ‘MCMC for continuous-time discrete-state systems’

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Proposition 1 The path (V, L, W) returned by algorithm 1 corresponds to a sample from the semi-Markov process parametrized by (π_0, A) .

Proof. Without any loss of generality, assume that the system has just entered state $s \in \mathcal{S}$ at time 0.

Suppose that t is the time of n th candidate jump, so that there were $n - 1$ rejected transitions on the interval $[0, t]$. Let these occur at times $(w_1, w_2, \dots, w_{n-1})$, with $t = w_n$. Recalling that these were generated from the hazard function $B_s(t)$, and letting $[n - 1]$ represent the set of integers $\{1, \dots, n - 1\}$, we have:

$$\begin{aligned}
 & P((w_1, \dots, w_n), \{v_i = s, l_i = (w_i - w_0) \forall i \in [n - 1]\}, v_n = s', l_n = 0 | w_0, v_0 = s) \\
 &= \left(\prod_{k=1}^{n-1} B_s(l_k) \exp \left(- \int_{l_{k-1}}^{l_k} B_s(\tau) d\tau \right) \left(1 - \frac{A_s(l_k)}{B_s(l_k)} \right) \right) \tag{1} \\
 & \quad \left(B_s(l_{n-1} + \Delta w_{n-1}) \exp \left(- \int_{l_{n-1}}^{l_{n-1} + \Delta w_{n-1}} B_s(\tau) d\tau \right) \left(\frac{A_{ss'}(l_{n-1} + \Delta w_{n-1})}{B_s(l_{n-1} + \Delta w_{n-1})} \right) \right) \\
 &= \exp \left(- \int_0^{l_{n-1} + \Delta w_{n-1}} B_s(\tau) d\tau \right) \left(\prod_{k=1}^{n-1} (B_s(l_k) - A_s(l_k)) \right) A_{ss'}(l_{n-1} + \Delta w_{n-1}) \tag{2}
 \end{aligned}$$

Integrating out w_1 to w_{n-1} (and thus l_1 to l_{n-1}), we have

$$\begin{aligned}
 & P(w_n = t, \{v_i = s \forall i \in [n - 1]\}, v_n = s', l_n = 0 | w_0 = 0, v_0 = s) \tag{3} \\
 &= \exp \left(- \int_0^t B_s(\tau) d\tau \right) A_{ss'}(w_n) \\
 & \quad \left(\int_0^t \int_{l_1}^t \dots \int_{l_{n-2}}^t \prod_{k=1}^{n-1} (B_s(l_k) - A_s(l_k) dl_k) \right) \\
 &= A_{ss'}(t) \exp \left(- \int_0^t B_s(\tau) d\tau \right) \frac{1}{(n-1)!} \left(\int_0^t d\tau (B_s(\tau) - A_s(\tau)) \right)^{n-1} \tag{4}
 \end{aligned}$$

The expression above gives the probability of transitioning from state s to s' after a wait of t time units, with $n - 1$ rejected candidate jumps. Summing out $n - 1$, we get the transition probability. Thus,

$$\begin{aligned}
 & P(s_{next} = s', t_{next} = t | s_{curr} = s, t_{curr} = 0) \\
 &= A_{ss'}(t) \exp \left(- \int_0^t B_s(\tau) d\tau \right) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\int_0^t d\tau (B_s(\tau) - A_s(\tau)) \right)^{n-1} \\
 &= A_{ss'}(t) \exp \left(- \int_0^t A_s(\tau) d\tau \right) \tag{5}
 \end{aligned}$$

This is the desired result. \square

Proposition 2 Conditioned on a trajectory (S, T) of the sMJP, the thinned events \tilde{W} are distributed as a Poisson process with intensity $B(t) - A(t)$.

Proof. We will consider the interval of time $[t_i, t_{i+1}]$, so that the sMJP entered state s_i at time t_i , and remained there until time t_{i+1} , when it transitioned to state s_{i+1} . Exploiting the independence properties of the sMJP and the Poisson process, we only need to consider resampling thinned events on this interval. Call this set of thinned events $\tilde{W} \equiv \{\tilde{w}_1, \dots, \tilde{w}_{n-1}\} \in [t_i, t_{i+1}]$, and call the corresponding set of labels $\tilde{V} \equiv \{\tilde{v}_1, \dots, \tilde{v}_{n-1}\}$ and $\tilde{L} \equiv \{\tilde{l}_1, \dots, \tilde{l}_{n-1}\}$ (to avoid notational clutter, we do not indicate that \tilde{W} and \tilde{L} are actually restrictions to $[t_i, t_{i+1}]$). Observe that each element of $\tilde{v}_j \in \tilde{V}$ equals s_i , while each element $\tilde{l}_j \in \tilde{L}$ equals $\tilde{w}_j - t_i$. We write this as $\tilde{V} = s_i$ and $\tilde{L} = \tilde{W} - t_i$. Then, by Bayes rule, we have

$$\begin{aligned}
& P(\tilde{W}, \tilde{V} = s_i, \tilde{L} = \tilde{W} - t_i | s_i, t_i, s_{i+1}, t_{i+1}) \tag{6} \\
&= \frac{P(\tilde{W}, \tilde{V} = s_i, \tilde{L} = \tilde{W} - t_i, v_n = s_{i+1}, w_n = t_{i+1}, l_n = 0 | v_0 = s_i, w_0 = t_i, l_0 = 0)}{P(s_{i+1}, t_{i+1} | s_i, t_i)} \\
&= \frac{\exp\left(-\int_{t_i}^{t_{i+1}} B(\tau) d\tau\right) \left(\prod_{k=1}^{n-1} (B(\tilde{w}_k) - A(\tilde{w}_k))\right) A_{s_i s_{i+1}}(t_{i+1} - t_i)}{A_{s_i s_{i+1}}(t_{i+1} - t_i) \exp\left(-\int_{t_i}^{t_{i+1}} A(\tau) d\tau\right)} \\
&= \exp\left(-\int_{t_i}^{t_{i+1}} B(\tau) - A(\tau) d\tau\right) \left(\prod_{k=1}^{n-1} (B(v_k) - A(v_k))\right)
\end{aligned}$$

This is just the density of a Poisson process on (t_i, t_{i+1}) with intensity $(B(t) - A(t))$, which is what we set out to prove. \square